

RIGOROUS CALCULUS PRACTICE

China-inspired

Worked solutions and marking guide

Parameter cases, root counting and inequality proofs

60 minutes | 3 structured questions | 30 marks

No calculator. Give exact answers and complete mathematical justifications.

This is a calculus-focused enrichment paper inspired by selected exam approaches. It is not a full national examination or a claim of syllabus equivalence. Europe has many different examination systems.

How to mark fairly

Award credit for mathematically equivalent methods. The point allocations are Academy practice rubrics, not official exam-board rubrics. Give method credit for sound work even if a later arithmetic error occurs; avoid penalizing the same error twice. Do not award a reasoning point for an unsupported correct number.

Add the marks across the three questions for a score out of 30. Use the topic labels to plan revision. No national grade conversion is implied.

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Structured solution 1

A logarithm with a parameter

Parameter analysis and root counting

10 marks

(a) [3] $f_a' = 1/x - a$ is positive for $0 < x < 1/a$ and negative for $x > 1/a$. The maximum occurs at $x = 1/a$ and equals $a - 1 - \ln a$.

Marking: 1 mark for the derivative and intervals; 1 for the maximum point; 1 for the maximum value.

(b) [3] Let $h(a) = a - 1 - \ln a$. Then $h' = 1 - 1/a$, so h has unique minimum $h(1) = 0$ and is strictly positive otherwise. Thus the maximum of f_a is at most zero exactly when $a = 1$.

Marking: 1 mark for analyzing h ; 1 for its equality case; 1 for both directions of the conclusion.

(c) [2] The function $\ln x - ax$ tends to $-\infty$ as $x \rightarrow 0^+$ and $x \rightarrow \infty$, and has unique maximum $m = -\ln a - 1$ at $1/a$. There are two positive solutions if $b < m$, one if $b = m$, and none if $b > m$.

Marking: 1 mark for the end behavior and maximum; 1 for all three root counts.

(d) [2] For $a = 1$, $f_1(1) = 0$ is the unique maximum and all other values are negative. The graph touches the axis at $(1, 0)$ without changing sign.

Marking: 1 mark for uniqueness; 1 for the no-crossing explanation.

Equivalent sound methods earn the same credit.

Structured solution 2

An absolute maximum for every parameter

Global extrema, parameter cases, exact algebra

10 marks

(a) [2] $g_a' = 3(x^2 - a)$. If $a \leq 0$, g_a is strictly increasing (with only a stationary point at 0 if $a = 0$), so $M = 1 - 3a$ at $x = 1$. If $a \geq 1$, it decreases on the interior, so $M = 3a - 1$ at $x = -1$.

Marking: 1 mark for each range with derivative reasoning.

(b) [3] For $0 < a < 1$, stationary points are $x = \pm\sqrt{a}$. The derivative signs are $+, -, +$, so $-\sqrt{a}$ is a local maximum of value $2a^{3/2}$, and $\sqrt{a}$ is a local minimum. Maximum candidates are $1 - 3a$, $3a - 1$, and $2a^{3/2}$.

Marking: 1 mark for the points; 1 for classification; 1 for all maximum candidates.

(c) [3] Write $u = \sqrt{a} \in (0, 1)$. Then $2a^{3/2} - (3a - 1) = 2u^3 - 3u^2 + 1 = (u - 1)^2(2u + 1) > 0$. Also $2a^{3/2} = 1 - 3a$ is $2u^3 + 3u^2 - 1 = (2u - 1)(u + 1)^2 = 0$, so $a = 1/4$.

Marking: 1 mark for the first factorization and sign; 1 for the equality equation; 1 for $a = 1/4$.

(d) [2] $M(a) = \begin{cases} 1 - 3a, & a \leq 1/4, \\ 2a^{3/2}, & 1/4 < a < 1, \\ 3a - 1, & a \geq 1. \end{cases}$ For $a < 1/4$ the maximizer is 1; for $a = 1/4$ both 1 and $-1/2$

maximize; for $1/4 < a < 1$ it is $-\sqrt{a}$; for $a \geq 1$ it is -1 .

Marking: 1 mark for the complete formula; 1 for all maximizing points, including the tie.

Equivalent sound methods earn the same credit.

Structured solution 3

The sharp constant in an inequality

Inequality proof and optimal constants

10 marks

(a) [2] For $p(x) = x - \ln(1+x)$, $p(0) = 0$ and $p' = x/(1+x) > 0$ for $x > 0$. Thus $p(x) > 0$.

Marking: 1 mark for the derivative; 1 for the monotonicity argument.

(b) [3] $q' = H(x)/x^3$. Direct differentiation gives $H' = -x^2/(1+x)^2 < 0$ and $H(0) = 0$. Therefore $H(x) < 0$ for $x > 0$, and so $q' < 0$.

Marking: 1 mark for q' ; 1 for H' ; 1 for the sign argument.

(c) [3] The proposed inequality is equivalent to $C \leq q(x)$. Since q decreases, its minimum on $(0, 1]$ is $q(1) = 1 - \ln 2$. This value works for every x in the interval, and any larger value fails at $x = 1$. Thus $C = 1 - \ln 2$.

Marking: 1 mark for equivalence; 1 for validity; 1 for sharpness.

(d) [2] L'Hospital gives $\lim q(x) = \lim [x/(1+x)]/(2x) = 1/2$. Strict decrease then gives $1 - \ln 2 < q(x) < 1/2$ for $0 < x < 1$.

Marking: 1 mark for the limit; 1 for both strict bounds.

Equivalent sound methods earn the same credit.