

RIGOROUS CALCULUS PRACTICE

Japan-inspired

Worked solutions and marking guide

Integral recurrences, symmetry and geometric proofs

60 minutes | 3 structured questions | 30 marks

No calculator. Give exact answers and complete mathematical justifications.

This is a calculus-focused enrichment paper inspired by selected exam approaches. It is not a full national examination or a claim of syllabus equivalence. Europe has many different examination systems.

How to mark fairly

Award credit for mathematically equivalent methods. The point allocations are Academy practice rubrics, not official exam-board rubrics. Give method credit for sound work even if a later arithmetic error occurs; avoid penalizing the same error twice. Do not award a reasoning point for an unsupported correct number.

Add the marks across the three questions for a score out of 30. Use the topic labels to plan revision. No national grade conversion is implied.

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Structured solution 1

An integral recurrence and a limit

Integral recurrence, bounds, sequence limits

10 marks

(a) [3] $I_0 = [\ln(1+x)]_0^1 = \ln 2$. Adding integrands gives $I_n + I_{n+1} = \int_0^1 x^n(1+x)/(1+x)dx = 1/(n+1)$.

Marking: 1 mark for I_0 ; 1 for combining integrands; 1 for the recurrence.

(b) [2] $I_1 = 1 - \ln 2$, $I_2 = 1/2 - I_1 = \ln 2 - 1/2$, and $I_3 = 1/3 - I_2 = 5/6 - \ln 2$.

Marking: 1 mark for the intermediate terms; 1 for I_3 .

(c) [3] $I_n - \frac{1}{2(n+1)} = \int_0^1 \frac{x^n(1-x)}{2(1+x)}dx$. The integrand is nonnegative and at most $x^n(1-x)/2$. Integrating that bound gives $\frac{1}{2}[1/(n+1) - 1/(n+2)] = 1/[2(n+1)(n+2)]$.

Marking: 1 mark for rewriting the difference; 1 for bounding; 1 for integrating the bound.

(d) [2] Multiply the inequality by $n+1$ to obtain $0 \leq (n+1)I_n - 1/2 \leq 1/[2(n+2)]$. The squeeze theorem gives the first limit $1/2$. Then $I_{n+1}/I_n = \frac{(n+2)I_{n+1}}{(n+1)I_n} \frac{n+1}{n+2} \rightarrow 1$.

Marking: 1 mark for each limit with reasoning.

Equivalent sound methods earn the same credit.

Structured solution 2

Oscillation and a volume invariant

Symmetry, substitution, geometric invariants

10 marks

(a) [2] Put $u = nx$. Then $\text{area} = \frac{1}{n} \int_0^{n\pi} |\sin u| du$. Each interval of length π contributes 2, so the total is $2n/n = 2$.

Marking: 1 mark for substitution; 1 for the period argument and area.

(b) [2] $\sin(n\pi - nx) = \sin(n\pi) \cos(nx) - \cos(n\pi) \sin(nx) = (-1)^{n+1} \sin(nx)$. Taking absolute values proves the identity.

Marking: 1 mark for the identity expansion; 1 for the absolute-value conclusion.

(c) [3] The substitution and symmetry give $J_n = \int_0^\pi (\pi - u) |\sin(nu)| du = 2\pi - J_n$. Thus $2J_n = 2\pi$ and $J_n = \pi$.

Marking: 1 mark for the substituted integral; 1 for $2\pi - J_n$; 1 for the exact value.

(d) [3] About the y -axis, $V_n = 2\pi J_n = 2\pi^2$. About the x -axis, disks give $\pi \int_0^\pi \sin^2(nx) dx = \pi [x/2 - \sin(2nx)/(4n)]_0^\pi = \pi^2/2$. Neither depends on positive integer n .

Marking: 1 mark for the shell volume; 1 for the disk integral; 1 for its value and independence.

Equivalent sound methods earn the same credit.

Structured solution 3

Two tangents to a parabola

Tangency, geometric proof, parameter elimination

10 marks

(a) [2] The tangent at a is $L_a(x) = 2ax - a^2$, and at b it is $L_b(x) = 2bx - b^2$. Equality gives $x = (a + b)/2$; substitution gives $y = ab$.

Marking: 1 mark for both tangents; 1 for the intersection.

(b) [2] $x^2 - L_a(x) = (x - a)^2 \geq 0$, with equality only at a . Similarly $x^2 - L_b(x) = (x - b)^2 \geq 0$.

Marking: 1 mark for each squared-difference argument.

(c) [4] Let $m = (a + b)/2$. Since $L_b - L_a = 2(b - a)(x - m)$, the upper tangent is L_a on $[a, m]$ and L_b on $[m, b]$. Thus area $= \int_a^m (x - a)^2 dx + \int_m^b (x - b)^2 dx = \frac{(m-a)^3 + (b-m)^3}{3} = (b - a)^3/12$.

Marking: 1 mark for the boundary switch; 1 for the split integral; 1 for integration; 1 for the formula.

(d) [2] Area 18 gives $(b - a)^3 = 216$, so $b - a = 6$. The midpoint condition gives $a + b = 2$, hence $a = -2$, $b = 4$, and $y = ab = -8$.

Marking: 1 mark for solving for a, b ; 1 for $y = -8$.

Equivalent sound methods earn the same credit.