

RIGOROUS CALCULUS PRACTICE

Japan-inspired

Calculus challenge mock

Integral recurrences, symmetry and geometric proofs

60 minutes | 3 structured questions | 30 marks

No calculator. Give exact answers and complete mathematical justifications.

This is a calculus-focused enrichment paper inspired by selected exam approaches. It is not a full national examination or a claim of syllabus equivalence. Europe has many different examination systems.

Before you begin

Write your name and date below. Keep the separate solutions file closed until you finish. Show equations, reasoning, units and required justifications. Use radian mode. Unless an exact answer is requested, round final numerical answers to three decimal places and retain unrounded values during calculations.

Name: _____ Date: _____

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Structured challenge

No calculator | 60 minutes for all three questions

Question 1: An integral recurrence and a limit

10 marks

For each integer $n \geq 0$, let $I_n = \int_0^1 x^n/(1+x) dx$. All answers must be justified without numerical integration.

(a) Find I_0 and prove $I_n + I_{n+1} = 1/(n+1)$. [3]

(b) Use the recurrence to find I_3 exactly. [2]

Show sufficient work to support your answers.

Structured challenge

No calculator | 60 minutes for all three questions

Question 1: An integral recurrence and a limit

10 marks

continued

Use the definitions and information on the preceding page.

(c) Prove $0 \leq I_n - 1/[2(n+1)] \leq 1/[2(n+1)(n+2)]$. [3]

(d) Deduce $\lim_{n \rightarrow \infty} (n+1)I_n$ and $\lim_{n \rightarrow \infty} I_{n+1}/I_n$. [2]

Show sufficient work to support your answers.

Structured challenge

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Question 2: Oscillation and a volume invariant

10 marks

For a positive integer n , let R_n be the region between $y = |\sin(nx)|$ and the x -axis on $0 \leq x \leq \pi$. Let V_n be its volume of revolution about the y -axis. You may use the cylindrical-shell formula $V = 2\pi \int x h(x) dx$, where h is the nonnegative shell height.

(a) Find the area of R_n and prove that it does not depend on n . [2]

(b) Show that $|\sin(n(\pi - x))| = |\sin(nx)|$. [2]

Show sufficient work to support your answers.

Structured challenge

No calculator | 60 minutes for all three questions

Question 2: Oscillation and a volume invariant

10 marks

continued

Use the definitions and information on the preceding page.

(c) Let $J_n = \int_0^\pi x |\sin(nx)| dx$. Use the substitution $u = \pi - x$ to find J_n exactly. [3]

(d) Find V_n . Also find the volume obtained by rotating R_n about the x -axis, and explain whether either volume depends on n . [3]

Show sufficient work to support your answers.

Structured challenge

No calculator | 60 minutes for all three questions

Question 3: Two tangents to a parabola

10 marks

Let $a < b$. Tangents to $y = x^2$ at $x = a$ and $x = b$, together with the parabolic arc between those points, enclose a finite region. Its lower boundary is the upper of the two tangent lines.

(a) Find the tangent equations and their intersection point. [2]

(b) Prove algebraically that each tangent lies on or below the parabola for every real x . [2]

Show sufficient work to support your answers.

Structured challenge

No calculator | 60 minutes for all three questions

Question 3: Two tangents to a parabola

10 marks

continued

Use the definitions and information on the preceding page.

(c) Prove that the area of the enclosed region is $(b - a)^3/12$. Your integral must account for the change in lower boundary. [4]

(d) The tangent intersection has x -coordinate 1 and the enclosed area is 18. Determine a , b , and the intersection's y -coordinate. [2]

Show sufficient work to support your answers.